International Journal of Applied Science and Mathematical Theory E- ISSN 2489-009X
P-ISSN 2695-1908, Vol. 11 No. 4 2025 www.iiardjournals.org online version

Crude Oil Price Modeling using Long Memory Processes: Inference and Applications

Wobo Omezuruike Gideon
Department of Statistics
Captain Elechi Amadi Polytechnic
Rumuola, Port Harcourt
Email: omezuruike.wobo@portharcourtpoly.edu.ng

Deebom Zorle Dum
Research Statistician,
Department of Mathematics & Computer Science,
Rivers State Universal Basic Education Board, Port Harcourt
Email: zorle.deebom1@ust.edu.ng
ORCID: 0000-0002-8086-0267
DOI: 10.56201/ijasmt.vol.11.no4.2025.pg31.51

Abstract
This study uses advanced statistical tests and models to investigate the presence of long-term memory in oil price benchmarks such as Average Oil, Brent Crude, Dubai Crude, and West Texas Intermediate. The data for this study are historical prices of major benchmark crude oils. For example, average benchmark crude oil prices obtained by taking a weighted average of these various crude oils (COA), Brent Crude (COB), Dubai Crude (COD) and West Texas Intermediate (WTI) are calculated. The data is taken from the Organization of the Petroleum Exporting Countries website (https://www.opec.org) and a total of 1984 data points are extracted for months from January 1982 to April 2023. ARFIMA, FIGARCH, HYGARCH, and FIAPARCH models are used to capture the complex dynamics and long-term dependencies of oil price fluctuations. Using the Akaike Information Criteria (AIC), the ARFIMA (1, -0.021, 1) model was considered the most appropriate among the competing models for Long Memory Processes in Crude Oil Prices. The results indicate significant long-term dependencies, persistence, and volatility accumulation in the oil price benchmarks. The results show that the crude oil price benchmarks exhibit asymmetry, non-stationarity, and long-run dependence, with significant fractional integral roots (d) and other parameters estimated by the model. The study concludes that shocks have transitory effects and that all series, except brent crude, exhibit mean-reverting behavior. The results have implications for oil price forecasting and modeling and highlight the importance of considering long memory processes in capturing the complex dynamics of oil price fluctuations.

Key Words: *Crude, Oil, Price Long, Memory Processes, Inference & Application*

1. Introduction

For a long time, oil prices have dominated attention in the global economy, having an impact on everything from international trade balances to rates of inflation. The fluctuations and unpredictability of oil prices carry substantial consequences for investors, governments, and sectors that rely on this essential resource (Hamilton, 2009). More advanced and reliable approaches must be developed due to the global oil market's complexity, volatility, and the shortcomings of current models (Kilian, 2009). To improve inference and support data-driven decision-making in the oil industry and related fields, this study explores the application of long-memory processes that can effectively capture the complex dynamics and long-term dependencies characteristic of oil price fluctuations (Deebom, Mazi, Chims, Richard & George,2021). This study attempts to solve the problem of accurately modeling and forecasting oil prices.

The long-term interdependence and enduring patterns present in oil price changes are frequently missed by traditional crude oil price modeling techniques, producing imprecise forecasts and less-than-ideal decision-making. The intricate dynamics and long-term relationships included in changes in oil prices are sometimes too complicated for traditional modeling techniques to adequately represent. This article provides a novel framework for comprehending and forecasting price fluctuations by investigating the use of long memory processes in modeling crude oil prices. This research attempts to deliver more accurate inference capabilities by utilizing the advantages of extended memory models, ultimately guiding data-driven decision making in the oil sector and beyond.

3. Methodology

The data for this study are historical prices of major benchmark crude oils such as the average benchmark crude oil price obtained by taking the weighted average of these various crude oils (COA), Brent crude oil (COB), Dubai crude oil (COD) and West Texas Intermediate (WTI) are calculated. The data is from the Organization of Petroleum Exporting Countries website (https://www.opec.org) and a total of 1984 data points were extracted for months from January 1982 to April 2023. The two robust statistical modeling and forecasting software programs used for the analysis were Stata and Oxmetrics. Time charts of the data series were also created to reveal the underlying dynamics of the crude oil price time series by making the trends and seasonality of the data easier to see. Similarly, descriptive statistics were examined to explore some characteristics of the data such as the mean, variance and skewness of the log returns. Unit root tests were performed to check for signs of non-stationarity in the data. The raw price data was converted into log returns, which show the rate of change in prices over time, and volatility, which measures the amount of fluctuation in prices. Time series modelling requires that the data are stationary, and this phase ensures that the log returns and volatility of the crude oil price benchmark were scaled to conditional compound monthly returns, calculated as follows:

$$COAR = Log\left(\frac{COA_t}{COA_{t-1}}\right) X \frac{100}{1} \tag{1}$$

The returns on average benchmark oil price which is calculated by taking a weighted average of these different crude oils (COAR), Brent Crude (COBR), Dubai Crude (CODR) and West Texas Intermediate (COWTIR).

Symmetric and Asymmetric Volatility in Long Memory Processes

When modeling financial time series such as crude oil prices, it is important to consider volatility and persistence (Deebom, Bharat & Inamete,2020). These properties are captured by long memory processes, which are characterized by slowly decreasing autocorrelation and infinite second moments. On the other hand, traditional long memory models such as ARFIMA, FIGARCH, and HYGARCH assume symmetric volatility, which is not always the case. Recent advances in long memory modeling have resulted in the introduction of asymmetric models such as FIAPARCH, which can capture both symmetric and asymmetric volatility. In this study, we investigate symmetric and asymmetric volatility in long memory processes, paying special attention to the specifications of ARFIMA, FIGARCH, HYGARCH, and FIAPARCH. This study reviews the conclusions, characteristics and applications of these models, highlighting their advantages when applied to modelling crude oil prices.

ARFIMA Model

The ARFIMA model, according to Sanusi *et al*. (2015) simply represent an acronym that stand for autoregressive fractional integrated moving average. The general form of an ARFIMA model as was defined as:

$$\phi(B)(1-L)^d Y_t = b(L^q)\varepsilon, \ \varepsilon_t \approx (0,\sigma_t^2) \quad (2)$$

$\phi(B)$ Is the Autoregressive Operator

b is the Moving Average Operator

d represents a fractional integration real number parameter,
L denotes the lag operator and ft is a white noise residual.
$(1 - L)^d$ stands for the fractional differencing lag operator.
Considering two cases of the "d", when d = 0 and d = 1. The "d" used in the model lies between Zero and one i.e $0 \le d \le 1$(Granger & Joyeaux ,1980)

FIGARCH Model

Baillie, Bollerslev, and Mikkelsen introduced the FIGARCH model in Chungs (1999) in 1996. A statistical model known as FIGARCH or Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity is used to describe the behavior of long-term memory under variability (Chung, 1999). The FIGARCH model is defined and redefined in Chung's (1999) specification as follows:

$$\emptyset(L)(1-L)^d(\varepsilon_t^2 - \sigma^2) = [1 - \boldsymbol{\beta}(L)](\varepsilon_t^2 - \sigma_t^2) \quad (3)$$

where σ^2 is the unconditional variance of ε_t If we retain the same definition of λ(L) as in (22), we can formulate the conditional variance as:

$$h_t = \sigma^2 + \{1 - [1 - \boldsymbol{\beta}(L)]^{-1} \boldsymbol{\Phi}(L)(1 - L)^d\}(\varepsilon_t^2 - \sigma^2) \quad (4)$$

$$h_t = \sigma_t^2 + (L)(\varepsilon_t^2 - \sigma^2)$$

where: h_t is the return at time t
σ_t^2 is the conditional volatility at time t
ε_t is the error term at time t
$\boldsymbol{\Phi}$ is a constant
α and β are parameters that capture the short-term dynamics of volatility
$\boldsymbol{\Phi}$(L) is a polynomial in the lag operator L, of order m-1, where m is the maximum of p and q
d is a parameter that captures the long memory behavior of volatility, with 0 < d < 1
$(1 - L)^d$ is the fractional differencing operator. Note that when d = 0, the FIGARCH model reduces to the standard GARCH model, and when d= 1, it reduces to the Integrated GARCH (IGARCH) model.

HYGARCH model

Hyperbolic GARCH is derived from Davidson (2004). The hyperbolic GARCH model (HYGARCH) extends the conditional variance of the FIGARCH model by introducing weights to the differential operators. The HYGARCH model can model the long memory property in the conditional volatility using hyperbolic convergence rates. The HYGARCH model (1, d,1) can be written as:

$$h_t = \omega + \left[1-(1-\beta L)^{-1}\lambda L\left\{1+\alpha(1-L)^d - 1)\right\}\right]\varepsilon_t^2 \quad (5)$$

Where $\omega > 0$, $\alpha \geq 0$, $\beta < 1$ and $0 \leq d \leq 1$

Davidson (2004) notes that the HYGARCH model can also be considered a more general version of the FIGARCH model, with its hyperbolic convergence rate and larger amplitude extreme values. In fact, the hyperbolic GARCH model can also be considered a more general version of the FIGARCH model, with its hyperbolic convergence rate and larger amplitude extreme values than the simpler IGARCH and FIGARCH models.

FIAPARCH (p, d, q) model

The FIAPARCH (p, d, q) model by Tse (1998) modifies the FIGARCH process to account for asymmetries. It combines the long memory property with asymmetric conditional volatility features. The asymmetries may be explained by the so-called "leverage effect" (Black (1976)). Tse (1998) proposes the FIAPARCH (p, d, q) model by extending the FIGARCH (p, d, q) model and the APARCH10- model function $\left(|\varepsilon t| - \gamma\varepsilon t\right)^{\delta}$ to capture the asymmetries and long memory features of conditional volatility. The FIAPARCH (*p, d, q*) model can be written as:

$$\sigma_t^{\delta} = \omega + \{1 - [\,1 - \boldsymbol{\beta}(\mathrm{L})]^{-1}\, \boldsymbol{\Phi}(\mathrm{L})(1 - \mathrm{L})^{d}\}(\,|\,\varepsilon_t\,|\, - \boldsymbol{\gamma}\varepsilon_t)^2 \quad (6)$$

where *y* is the leverage coefficient, and δ is the parameter for the power term that takes (finite) positive values. When d = 0, the FLAPARCH (*p,d.q*) process reduces to APARCH of' Ding et al. (1993). When *y* = 0 and δ = 2, the process in Chung (1999) redefines the FIGARCII model as $\emptyset$ (L) $(1 - \mathrm{L})^d(\varepsilon_t^2 - \sigma^2)$ = [$1 - \boldsymbol{\beta}$(L)] $(\varepsilon_t^2 - \sigma_t^2)$, where σ^2 is the unconditional variance of ε_t reduces to the FIGARCH (*p,d,q)* specification. which includes Bol1erslev's1986) model when *d = 0,* and the integrated specification when *d = 1*, as special cases.

3.5. Steps involved in ARFIMA and FIGARCH Modeling

The following are the ARFIMA model's analysis steps:

1. Descriptive Statistics

In the analysis phase, we use descriptive statistics to investigate the characteristics and normality attributes of the historical benchmarks of crude oil prices and earnings. Chinyere et al. (2015) state that the test statistic obtained from the definition of the Jarque-Bera test indicates a joint test of skewness and kurtosis, which examines whether the data values have the characteristics of a normal distribution. The degrees of freedom of the test statistic under the null hypothesis of normal distribution is 2.

2. Determining ACF and PACF

Autoregressive conditional heteroskedasticity (ARCH) is further performed on the log transformation and is designed to measure the effect of heteroskedasticity, i.e., the time-varying variance in the series. This can be seen by plotting the ACF of the squared residuals from the model. If the autocorrelation function (ACF) of the squared residuals shows long-run dependence, then the ARCH effect is present. Similarly, to identify the components of the model, an analysis of the autocorrelation function (ACF) and partial autocorrelation function (PACF) of the original time series was required. Recording the ACF and PACF of the time series is the first step in determining the AR components. If the ACF shows a rapidly decreasing pattern where the initial values deviate significantly from zero, then the time series contains an AR component. It is equivalent to plotting the partial autocorrelation function (PACF) of the original time series. A sharp increase at lag 1 in the PACF followed by a sharp decrease to a value near zero indicates the presence of an MA component (1) in the time series. A large increase at lags 1 and 2 in the PACF indicates the presence of an MA component (2).

3. Long-term Memory Tests

Several tests, such as Lo's Rescaled Range (R/S) statistic, Geweke-Porter-Hudak (GPH) test, and Generalized Portmant statistic (GPS), are applied to time series to determine the presence of this long-term effect. Lo's Rescaled Range (R/S) statistic was originally proposed by Hurst (1951) and later modified by Lo (1991). Lo (1991) pointed out that the original statistic is not robust to short-range dependence. In addition, Geweke and Porter-Hudak (1983) proposed a semiparametric approach to test for long-term memory using the following regression: The Gaussian semiparametric estimation proposed by Robinson and Henry (1999) is based on the maximum likelihood estimator with Whittle approximation. The GPH estimation approach is used to determine whether the data has a long-term memory effect. When the value of d is between 0 and 1, as determined by the GPH estimation method, the data exhibits long memory effects.

4. Formulation of ARFIMA, FIGARCH, HYGARCH, FIAPARCH Model Structures

The formation of ARFIMA, FIGARCH, HYGARCH, FIAPARCH models is done by testing the data for stationarity and the presence of autocorrelation using the L Jung-Box (Q) test statistic. The presence of autocorrelation justifies identifying the AR and MA structures of the models using the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) plots. Based on the ACF and PACF charts, select one or more AR and MA in the ARFIMA, FIGARCH, HYGARCH, and FIAPARCH model structures. Next, the parameters of the ARFIMA, FIGARCH, HYGARCH and FIAPARCH models are calculated.

Results and Discussion

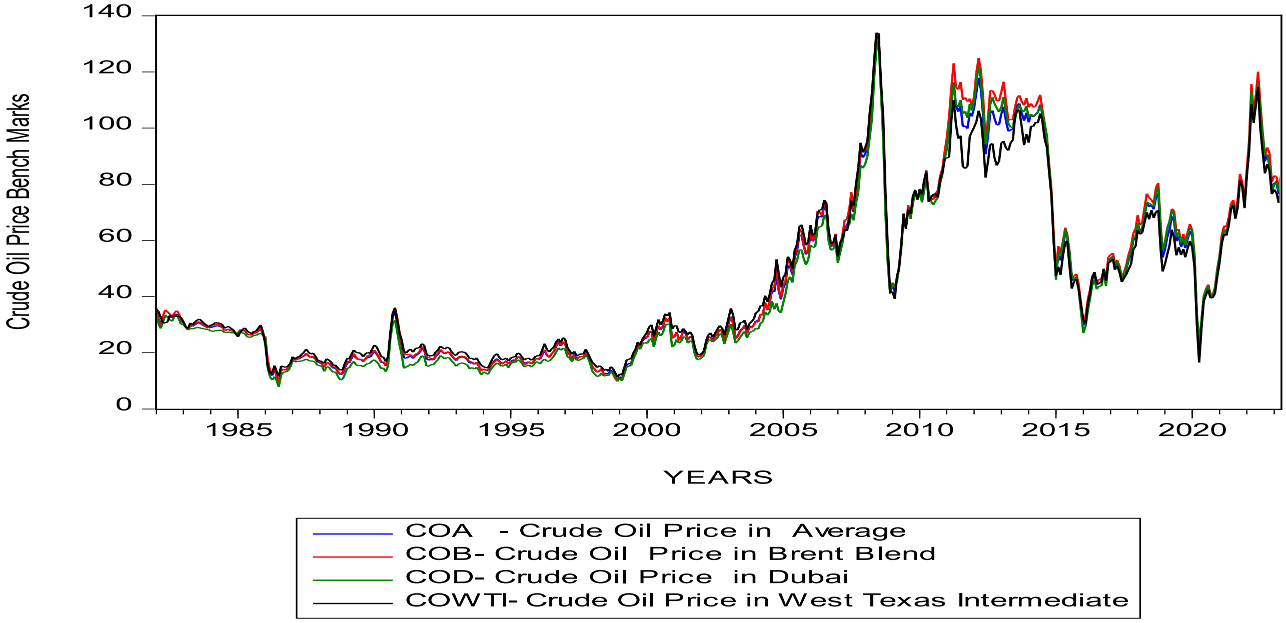

Figure 1 Time Plot on Raw Crude Oil Price Benchmarks

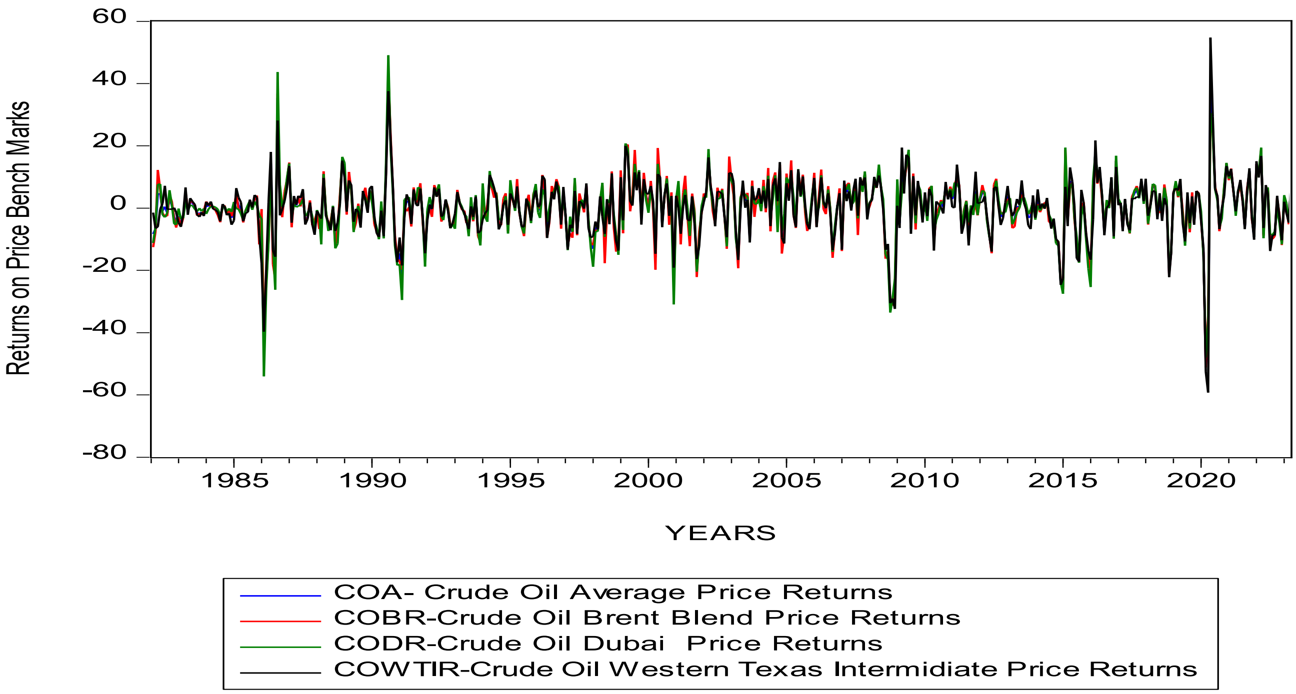

Figure 2 Time Plots on Crude Oil Price Benchmarks Returns

The time plots in Figures 1 to 2 are on the price and returns on crude oil price Average, Brent, Dubai and West Texas Intermediate from 1982, January to May 2023. The Figure 1 is done to visualized as well determine the trends in the movement in crude oil price benchmarks. Also, from visual examination of the crude oil prices behavior in figure 2 shows that the price series are stationary and fluctuate around the origin. It also reveals the presence of a clustering volatility.

Table 1: Descriptive Statistics for Crude Oil Price Benchmarks (Returns and Raw)

	COAR	COBR	CODR	COWTIR	COA	COB	COD	COWTI
Mean	0.173	0.172	0.182	0.164	44.885	46.0149	43.899	44.741
Median	0.902	0.565	0.866	0.801	30.700	30.975	28.950	31.730
Maximum	43.020	43.263	49.102	54.744	132.830	133.870	131.220	133.930
Minimum	-50.491	-51.143	-54.012	-59.262	9.620	9.450	7.850	11.310
Std. Dev.	9.163	9.3701	9.474	9.411	30.395	31.741	31.162	28.463
Skewness	-0.677	-0.549	-0.707	-0.676	0.890	0.9149	0.903	0.876
Kurtosis	8.313	6.859	9.274	11.028	2.603	2.657	2.612	2.648
Jarque-Bera	620.109	331.795	853.059	1367.020	68.692	71.623	70.548	65.952
Probability	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Observations	495	495	495	495	496	496	496	496

The results in Table 1 show the descriptive statistics of the returns and crude oil series relative to the Nigerian Crude Oil Price. This is done to check if the price benchmark returns meet the assumption of normality and is estimated using the Jacque-Bera test. The results show that the average prices of both yield and crude oil series are positive such that the average prices of Crude Oil are (0.173), Brent (0.172), Dubai Crude (0.182), West Texas Intermediate (0.164), and the average prices of West Texas Intermediate (44.885), Brent (46.0149), Dubai Crude (43.899), and West Texas Intermediate (44.741). Similarly, the skewness results show that the return series is on average (-0.677), Brent (-0.549), Dubai Crude (-0.707), and West Texas Intermediate (-0.676). This means that the series are all skewed to the right. On the other hand, the Crude Oil series has the Mean Price (0.890), Brent Crude Oil (0.9149), Dubai Crude Oil (0.903), and West Texas Intermediate Price (0.876), which are all skewed to the right. Kurtosis also indicates the number of tails in the data distribution. Thus, the series returns show that the mean (8.313), Brent (6.859), Dubai (9.274), and West Texas Intermediate (11.028) are all greater than 3, indicating that the data distribution is leptokurtic. The kurtosis of all raw series is less than 3: mean (2.603), Brent (2.657), Dubai (2.612), and West Texas Intermediate (2.648). This indicates that the data distribution is leptokurtic. Although the data set is focused on the tip and has many outliers, the reduction in kurtosis corresponds to the broadening of the tip and the "thickness" of the tail. The distribution of the data has a sharp peak. Similarly, the Jarque-Bera test is a goodness-of-fit test that determines whether the skewness and kurtosis of a data sample are consistent with a normal distribution. The results of the Halké-Bella statistic show that for the return series, such returns apply to the mean (620.109), Brent (331.795), Dubai (853.059), and West Texas Median Price (1367.020) whereas for the raw series mean (68.692), Brent (71.623), Dubai (70.548), and West Texas Median Price (65.952). The conclusion of the study is that since all p-values are less than 0.05, we reject the null hypothesis and conclude that all variables are not actually significant at the 95% confidence level. In this sense, Deebom and Tuaneh (2019), Deebom, Bharat, and Inamete (2020), Deebom, Ette and Nwikorga (2021) and Ali, Nzotta, Akujuobi, and Nwaimo (2022) suggest that alternative inferential statistics in testing the performance of partially integrated models with applications to price-earnings ratios include estimating unit roots, model identification using ACF and PACF, and determining the presence of long memory, which are performed before estimating the model parameters.

Table 2: Unit Test Results using the Augmented Dickey Fuller for Crude Oil Price Benchmarks

Variable	t-Statistic	P-Value	Remarks
COA	-2.220	0.199	**1(1)**
D(COA)	-14.976	0.000	
COB	-2.172	0.217	**1(1)**
D(COB)	-15.415	0.000	
COD	-2.171	0.217	**1(1)**
D(COD)	-14.637	0.000	
COWTI	-2.367	0.152	**1(1)**
D(COWTI)	-15.396	0.000	

The results of the unit test Results using the Augmented Dickey Fuller (ADF) for crude oil price Benchmarks **is** shown in Table 2. The variables of crude oil price Benchmarks were found to stationary at first difference. The results for test of the presence of long memory in crude oil price benchmarks were further investigated using Lo's R/S, GPH and Robinson Test as shown in Table 3 below.

Table 3: Results for Test of the Presence of Long Memory in Crude Oil Price Benchmarks

Raw Series	COA	COB	COD	COWTI
Lo's R/S Test	[0.861, 1.747]	[0.861, 1.747]	[0.861, 1.747]	[0.861, 1.747]
	[0.809, 1.862]	[0.809, 1.862]	[0.809, 1.862]	[0.809, 1.862]
	[0.721, 2.098]	[0.721, 2.098]	[0.721, 2.098]	[0.721, 2.098]
GPH Test				
$M = T^{0.5}$	[0.838***]	[0.873***]	[0.864***]	[0.785***]
$M = T^{0.6}$	[0.934***]	[0.989***]	[0.952***]	[0.865***]
$M = T^{0.7}$	[1.010***]	[1.015***]	[1.005***]	[0.954***]
$M = T^{0.8}$	[1.141***]	[1.104***]	[1.140***]	[1.127***]
Robinson Test				
0.5	[0.830***]	[0.881***]	[0.865***]	[0.788*]
0.6	[0.931***]	[0.956***]	[0.950***]	[0.862*]
0.7	[1.001***]	[1.006***]	[0.996***]	[0.946*]
0.8	[1.105***]	[1.069***]	[1.105***]	[1.093*]

The results were all tested at 1%, 5% and 10% level of Significance, while*, **, and *** represents the 1%, 5% and 10% level of significance.

The results in Table 3 are used to test for the presence of long-term memory in the crude oil price benchmark using Lo's R/S, GPH and Robinson tests. Lo's R/S test measures the fractal dimension of the time series and detects long-range dependence (LRD), while the GPH test provides statistics to detect LRD in non-stationary time series and Robinson test is used to detect asymmetry in the time series (Deebom, Essi & Amos, 2022). Volatility, which is an important feature of long-term memory processes. The inference drawn from the investigations is that since all the p-values are values less than 1, 5 and 10 percent level of Significance respectively, therefore we reject the null hypothesis and conclude that the series have long-range dependence or long memory, the GPH test results show the LRD is non-stationary and it is asymmetric in nature. The results for L Jung-Box (Q) Statistic are shown in table 5 below.

Table 4: Estimates of the L Jung-Box (Q) Statistic in Crude Oil Price Benchmarks

Prices		Q-statistic	Probability
COA	Q(5)	1592.717	0.020
	Q(10)	2922.286	0.000
	Q(20)	5109.223	0.000
	Q(50)	8984.009	0.000
COB	Q(5)	1584.439	0.000
	Q(10)	2901.609	0.000
	Q(20)	4932.883	0.000
	Q(50)	8082.934	0.000
	Q(5)	1622.64	0.000
COD	Q(10)	3008.647	0.000
	Q(20)	5106.029	0.000
	Q(50)	7404.828	0.000
COWTI	Q(10)	3008.647	0.000
	Q(20)	5106.029	0.000
	Q(50)	7404.828	0.000

The results were all tested at 1%, 5% and 10% level of significance respectively

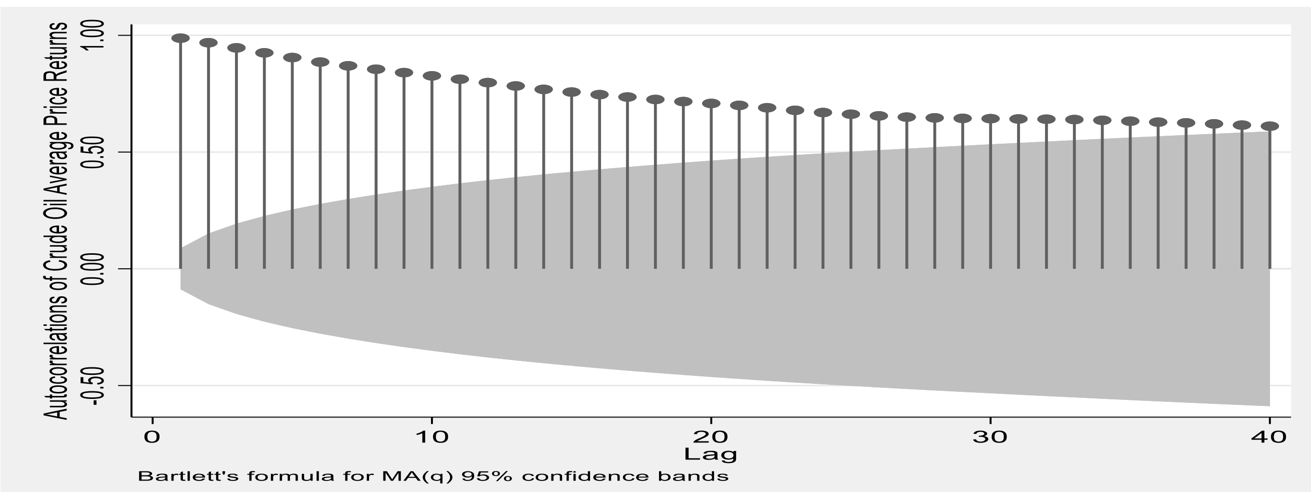

Figure 3: ACF on Crude Oil Average Price

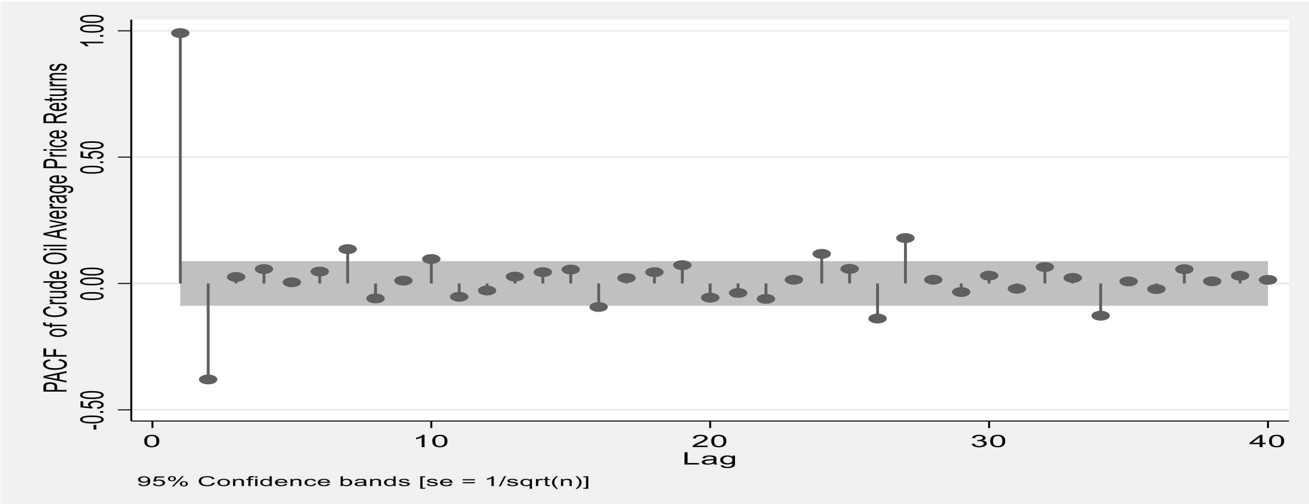

Figure 4: PACF for Crude Oil Price in Brent Blend

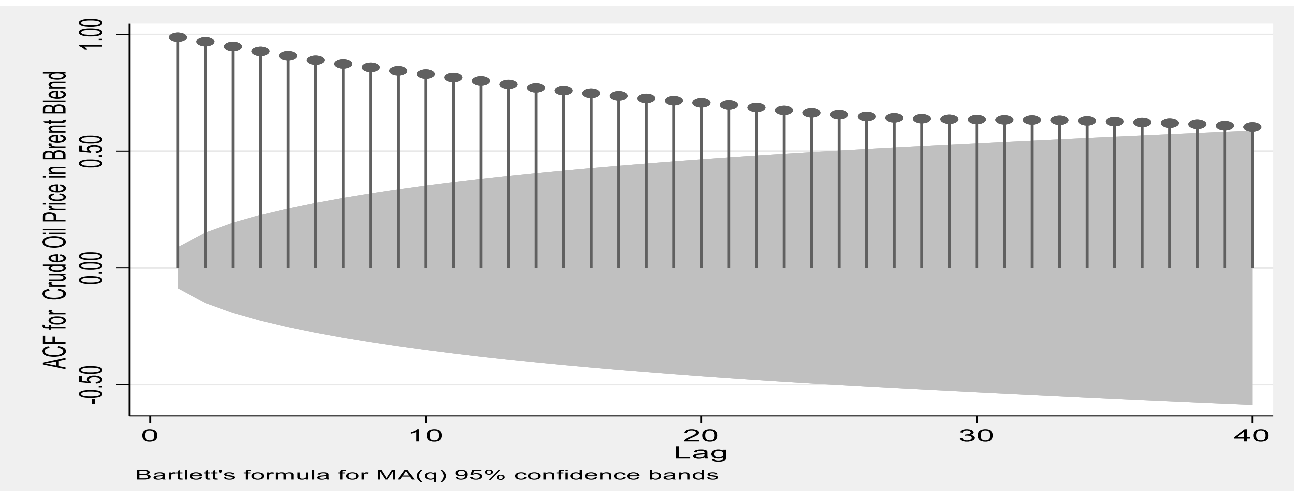

Figure 5: ACF for Crude Oil Price in Brent Blend

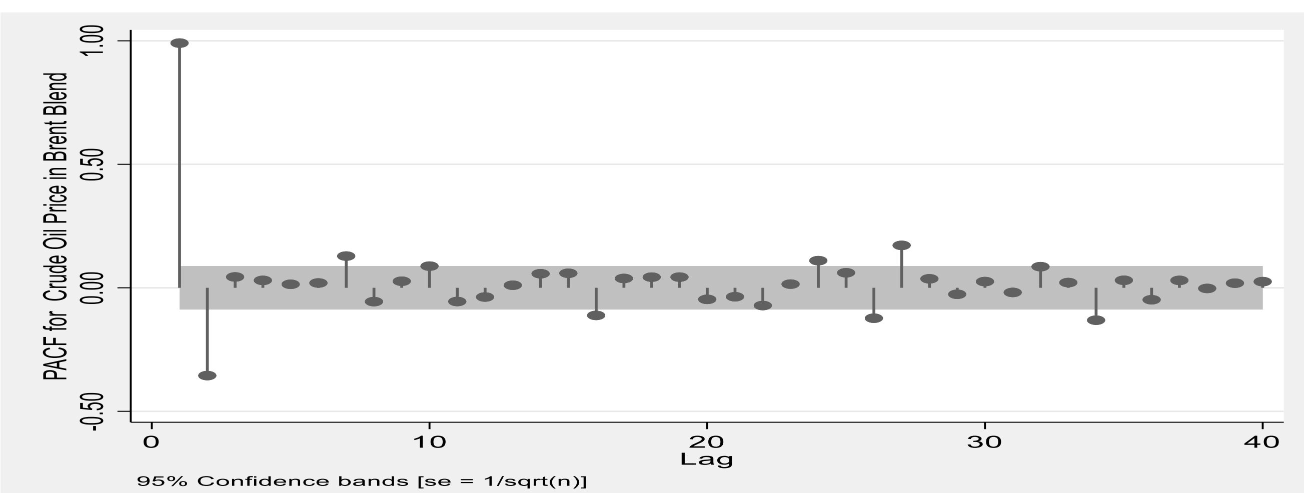

Figure 6: PACF for Crude Oil Price in Brent Blend

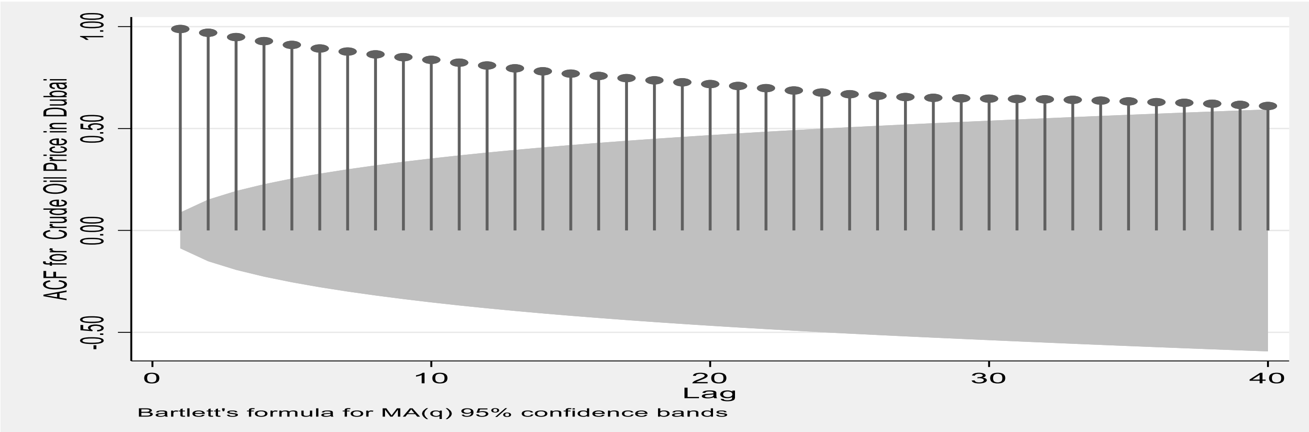

Figure 7*:* ACF for Crude Oil Price in Dubai

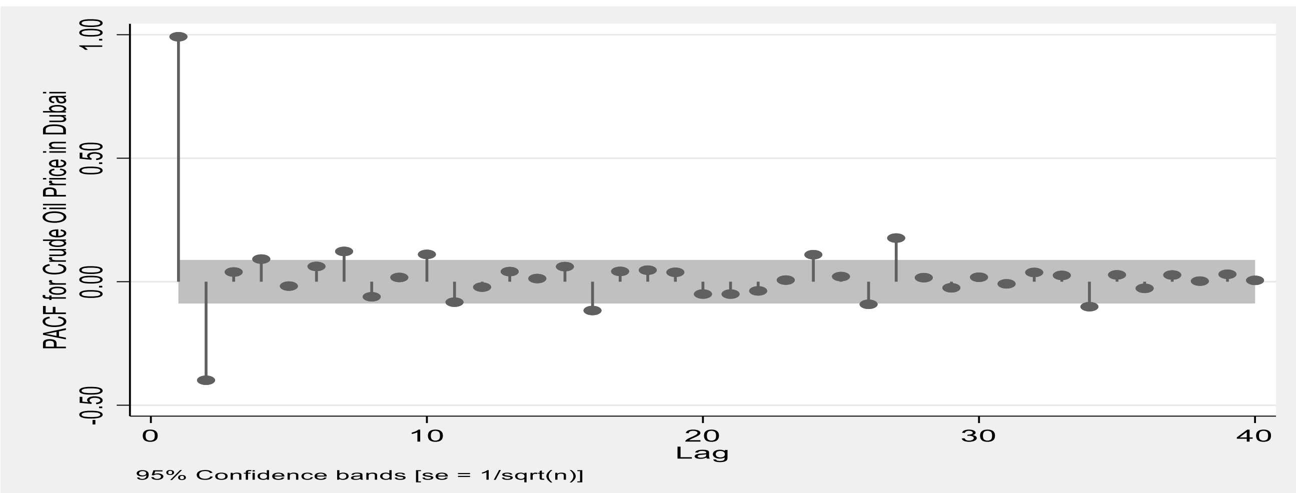

Figure 8*:* PACF for Crude Oil Price in Dubai

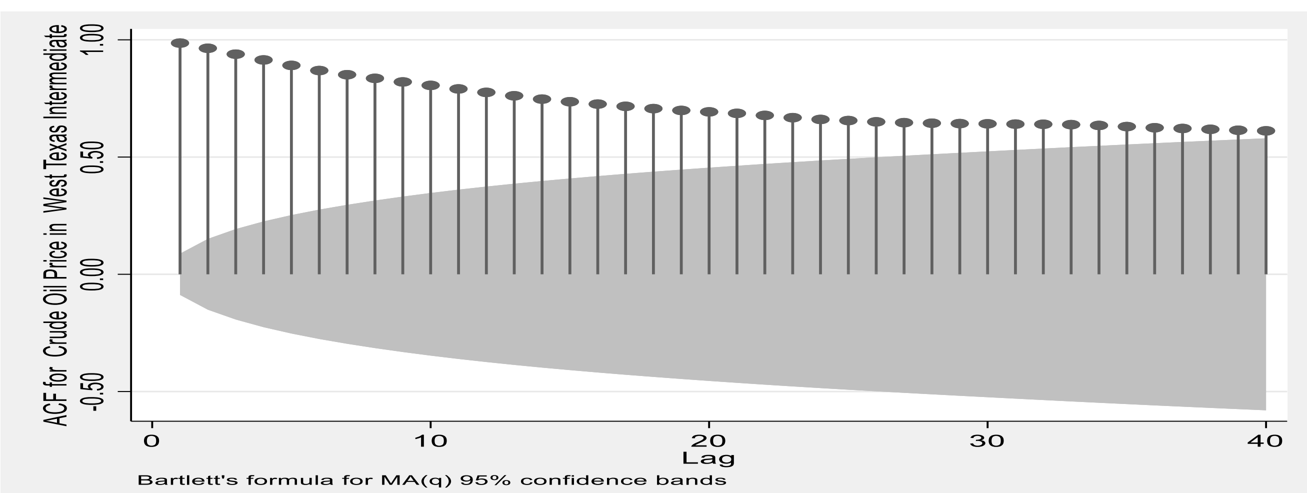

Figure 9*:* ACF for Crude Oil Price in West Texas Intermediate

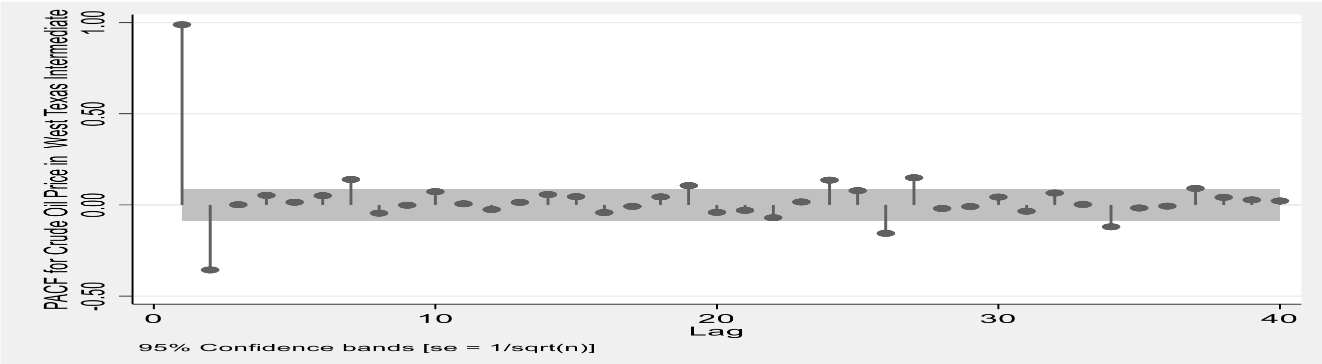

Figure 10*: PACF for Crude Oil Price in West Texas Intermediate*

In Figures 3-10, we have ACF and PACF of Crude Oil Prices, Figures 3 and 4 show the ACF (Autocorrelation Function) and PACF (Partial Autocorrelation Function) of the average crude oil price time series. Figures 5 and 6 show the ACF and PACF of the Brent Blend crude oil price time series, Figures 7 and 8 show the ACF and PACF of the Dubai crude oil price time series and Figures 9 and 10 show the ACF and PACF of the West Texas Intermediate (WTI) crude oil price time series. The plots of the ACF and PACF of Crude Oil Prices in Figures 3-10 shows a gradually declined with spikes 1, -2,7,10, -16,24, -26,27, and -34 above confidence band for the Autoregressive (AR) Model while the PACF falls from 1,2,4, 5 to 40 for the Moving Average Model. The Summary of Autoregressive (AR) and Moving Average (MA) lags Structure of the Models Using PACF and ACF Plots is shown in Table 5 below.

Table 5: Summary of Autoregressive (AR) and Moving Average (MA) LAGS Structure of the Models Using PACF and ACF Plots.

	PACF FOR Autoregressive (AR) Model	**ACF FOR Moving Average (MA) Model**
Variables	**LAGS**	**LAGS**
COA	1,-2,7,10,-16,24,-26,27,-34	1,2,4, 5,……………………………,40
COB	1,-2,7,10,-16,24,-26,27,-34	1,2,4, 5,……………………………,40
COD	1,-2,7,10,-16,24,-26,27,-34	1,2,4, 5……………………………,40
COWTI	1,-2,7,10,-16,24,-26,27,-34	1,2,4, 5,…………………………….,40

The results in Table 5 are the summary of Autoregressive (AR) and Moving Average (MA) lags structure of the models Using PACF and ACF Plots. From the results obtained, the lags for the structures for our models is considered between -26 to 27 for the Autoregressive (AR) Model while Moving Average (MA) lags lie between 1 to 40. However, to avoid overparameterization the considered lag one alone. This follows the results of ARFIMA model estimation for Crude Oil Price Benchmarks as shown in Table 6 below.

Table 6: Results of ARFIMA model estimation for Crude Oil Price Benchmarks

Raw Sales	**Models**	**Parameters**							**Least AIC**
		α	**Φ**	**β**	**D**	**ξ**	**AIC**	**BIC**	
COA	ARFIMA(1, 0.43,1)	49. 722 (9.128)	0.849 (0.000)	0.011 (0.889)	0.431 (0.000)	20.695 (0.000)	2076.37	2076.37	
COB	ARFIMA(1,-0.021,1)	1.917 (0.000)	0.984 (0.000)	0.236 (0.003)	-0.021 (0.790)	0.012 (0.000)	-531.204	-511.943	ARFIMA (1,-0.021,1)
COD	ARFIMA(1, 0.075,1)	171.640 (0.120)	0.995 (0.000)	0.249 (0.000)	0.075 (0.214)	60.156 (0.000)	2429.254	2448.512	
COWTI	ARFIMA(1, 0.075,1)								

The results were all tested at 1%, 5% and 10% level of Significance

The results in Table 6 is the ARFIMA model estimation for crude oil price benchmarks. The results show that the AR Model and MA are all positive and significantly different from zero at 1, 5 and 10 percent level of significance. The ranges of values of (d) lies between -0.021 and 0.431. The parameter (d) estimates are all significantly different from zero at 1, 5 and 10 percent level of significance except for crude oil price benchmark in brent whose **is** -0.021. These indicate that there is the present of long-range dependence; the fractionally integrated roots (d) are anti-persistence except for crude oil price benchmark in brent**.** The inference drawn from the investigations is that since an economic implication suggesting that shocks have a short-lived impact, with the crude oil price benchmark series exhibiting mean-reverting behavior except the case crude oil price benchmark in brent.

Table 7: Results for FIGARCH model estimation for Crude Oil Price Benchmarks

VARIABLES	**COA**	**COB**	**COD**	**COWTI**
Model Parameters	FIGARCH (1, d,1)	FIGARCH (1, d,1)	FIGARCH(1,d,1)	FIGARCH(1,d,1)
CST(M)	-0.009 (0.940)	-0.050 (0.701)	0.015 (0.898)	0.013 (0.925)
AR(1)	0.274 (0.110)	0.272 (0.165)	0.238 (0.050)	0.354 (0.090)
MA(1)	0.188 (0.100)	0.211 (0.027)	0.207 (0.064)	0.126 (0.363)
CST(V)	1.232 (0.517)	0.782 (0.433)	0.161 (0.000)	0.231 (0.824)
d-FIGARCH	0.580 (0.000)	0.564 (0.000)	1.450 (0.000)	0.484 (0.000)
ARCH(α_1)	-0.986 (0.000)	-0.987 (0.000)	-0.015 (0.970)	-0.993 (0.000)
GARCH (β_1)	-0.983 (0.000)	-0.975 (0.000)	0.909 (0.000)	-0.986 (0.000)
Loglikelihood	-1430.18	-1823.05	-200.82	-200.82
Means (μ)	51.097	1.009	1.948	1.948
Skewness	-1.27	-0.907	0.432	0.432
Kurtosis	0.70	-1.027	2.029`	2.029
Jarque-Bera	100.47	62.818	22.918	22.918

AIC	-5.217	5.450	5.234	5.383
SIC	5.285	5.509	5.293	5.442

The results were all tested at 1%, 5% and 10% level of Significance

The results in Table 7 contain the FIGARCH model estimation for crude oil price benchmarks. The results show that the AR Model and MA are all positive but not significantly different from zero at 1, 5 and 10 percent level of significance. The ARCH and GARCH model components are all negative and significantly different from zero at 1, 5 and 10 percent level of significance except for FIGARCH model estimation using crude oil price benchmark in Dubai whose ARCH (α_1)model is negative and not significantly different from zero at 1, 5 and 10 percent level of significance. The crude oil price benchmark in Dubai' volatility is mainly driven by long-term factors, and short-term clustering is not significant. The model can be simplified by omitting the ARCH component, reducing the number of parameters to estimate since the ARCH component does not contribute significantly to the volatility process.

Also, d-FIGARCH fractionally integrated roots (d) are negative and significantly different from zero at 1, 5and 10 percent level of significance. The ranges of values of (d) are less than zero (0.580, 0.564, 0.484) except for FIGARCH model estimation using crude oil price benchmark in Dubai whose fractionally integrated root (d) (d-FIGARCH) parameter is 1.450. The inference for the d-FIGARCH parameter to be less than zero indicates anti-persistence, or an asymmetric reaction to shocks. This means that the series tends to reverse itself. The d-FIGARCH parameter in FIGARCH model estimation using crude oil price average, brent and West Texas Intermediate benchmarks respond asymmetrically to shocks, there is short-term dependencies in the series, volatility tends to revert to its mean level quickly after a shock, and impact of past events on current volatility is limited. For the d-FIGARCH parameter (1.450) in FIGARCH model estimation using crude oil price benchmark in Dubai respond to shocks volatility remain high for an extended period after a shock. Also, the impact of past events on current volatility can be significant, even if they occurred far in the past.

Table 8: Results for HYGARCH model in Crude Oil Price Benchmarks

VARIABLES	**COA**	**COB**	**COD**	**COWTI**
Model Parameters	HYGARCH (1,d,1)	HYGARCH (1,d,1)	HYGARCH (1,d,1)	HYGARCH (1,d,1)
CST(M)	0.059 (0.612)	-0.084 (0.509)	0.128 (0.618)	0.686 (0.070)
AR(1)	0.189 (0.190)	0.188 (0.004)	0.132 (0.700)	0.012 (0.971)
MA(1)	0.196 (0.092)	0.252 (0.004)	0.260 (0.918)	0.181 (0.577)
CST(V)	-0.677 (0.276)	-3.167 (0.000)	-0.026 (0.918)	42.612 (0.010)
d-FIGARCH	0.580 (0.000)	0.445 (0.000)	0.702 (0.319)	0.611 (0.000)
ARCH(α_1)	-0.662 (0.000)	-0.962 (0.000)	0.841 (0.027)	-0.786 (0.000)
GARCH (β_1)	-0.757 (0.000)	-0.956 (0.000)	0.907 (0.000)	-0.822 (0.000)

Loglikelihood	0.646	0.397	0.370	-0.293
Means (μ)	0.268	0.276	0.277	0.164
Skewness	5.358	5.145	5.674	-0.676
Kurtosis	88.013	84.408	94.049	11.028
Jarque-Bera	6.978	52.46	6694.6	22.918
AIC	5.817	5.332	5.147	6.978
SIC	5.254	5.400	5.215	7.046

The results were all tested at 1%, 5% and 10% level of Significance

The results in Table 8 contain the HYGARCH model estimation for crude oil price benchmarks. The results show that the AR Model and MA are all positive and significantly different from zero at 1, 5 and 10 percent level of significance except for HYGARCH model estimation using crude oil price benchmark in Dubai and West Texas Intermediate not significantly different from zero at 1, 5 and 10 percent level of significance. The significance of all AR and MA parameters suggests that using crude oil price average, brent and Dubai have long memory, meaning that past shocks and patterns continue to influence current and future behavior. For crude oil price benchmark in Dubai and West Texas Intermediate, it suggests that they are largely driven by random fluctuations. Also, the ARCH and GARCH model components are all negative and significantly different from zero at 1, 5 and 10 percent level of significance except for HYGARCH model estimation using crude oil price benchmark in Dubai that has positive coefficient. This suggests that the volatility crude oil price average, brent and West Texas Intermediate is dampened. This means that crude oil price average, brent and West Texas Intermediate has the tendency to revert to its mean level with their volatility decreasing over time. Also, for the ARCH and GARCH model components in HYGARCH model estimation using crude oil price benchmark in Dubai to have had positive and significant coefficient, its exhibits volatility amplification which is highly sensitive to past shocks and events. The d-FIGARCH parameters in HYGARCH model estimation are all positive and significantly different from zero at 1, 5 and 10 percent level of significance respectively except for crude oil price benchmark in Dubai whose estimate is not significant. The positive d-FIGARCH parameter indicates that crude oil price average, brent and West Texas Intermediate has persistent volatility, meaning that volatility tends to persist over time while the reverse is the case for crude oil price benchmark in Dubai

Table 9: Results for FIAPARCH model in Crude Oil Price Benchmarks

VARIABLES	**COA**	**COB**	**COD**	COWTI
Model Parameters	FIAPARCH (1, d, 1)	FIAPARCH (1, d, 1)	FIAPARCH (1, d, 1)	FIAPARCH (1, d, 1)
CST(M)	0.059 (NA)	0.006 (0.957)	0.141 (0.205)	0.105 (0.507)
AR(1)	0.227 (0.190)	0.140 (0.372)	0.114 (0.302)	0.184 (0.144)
MA(1)	0.198 (0.092)	0.236 (0.003)	0.251 (0.002)	0.191 (0.091)
CST(V)	4.722 (0.276)	-1.779 (0.837)	92.271 (0.741)	3.376 (0.363)
d-FIGARCH	0.881 (0.000)	0.794 (0.000)	0.666 (0.002)	0.488 (0.000)

ARCH(α_1)	-3.335 (0.000)	-0.159 (0.385)	-0.006 (0.971)	-0.008 (0.965)
GARCH (β_1)	0.000 (0.000)	0.006 (0.586)	-0.002 (0.711)	0.001 (0.664)
APARCH(Gamma1)	-0.175 (0.088)	-0.262 (0.088)	-0.290 (0.001)	-0.223 (0.316)
APARCH(Delta)	4.141 (0.088)	3.509 (0.000)	3.215 (0.000)	3.165 (0.000)
Loglikelihood	-12672.534	-1296.662	-1270.715	-1281.162
Means (μ)	0.268	0.276	0.277	**0.251**
Skewness	5.358	5.145	5.674	**4.790**
Kurtosis	88.213	83.409	94.049	**77.648**
Jarque-Bera	6.978	3736.8	3736.8	22.918
AIC	5.138	5.275	5.275	**5.213**
SIC	5.214	5.352	5.352	**5.289**

The results were all tested at 1%, 5% and 10% level of Significance

The results in Table 9 contain the FIAPARCH model estimation for crude oil price benchmarks. The results show that the AR Models of the FIAPARCH models are all positive and not significantly different from zero at 1, 5 and 10 percent level of significance. This means that crude oil price benchmarks are not reversal such that previous returns do have a negative impact on the current returns. Also, MA components of the FIAPARCH model are all positive and significantly different from zero at the one percent level of significance. This simply means previous errors associated crude oil price benchmark have positive impact on the current returns and they volatility clustering. The d-FIGARCH parameters in FIAGARCH model estimation are all positive and significantly different from zero at 1, 5 and 10 percent level of significance respectively. This shows that long memory exists in all the crude oil price benchmarks. Also, for the ARCH of the FIAPARCH models are not significantly different from zero except for crude oil price benchmark average that is negative and significantly different from zero. This means there is a reversal in volatility clustering, where periods of high volatility are followed by periods of low volatility. GARCH model components in FIAPARCH model estimation for crude oil price benchmarks are not significant. This means that volatility doesn't persist over time. The APARCH(Gamma1) coefficients for average , Brent, and Dubai crude oil price benchmarks respectively are -0.17 ,- 0.262 and -0.290 are significant at 5 percent level while that of West Texas Intermediate is -0.223 is not significantly different from zero. This means that Brent, and Dubai crude oil price benchmarks negative shocks have a greater impact on volatility. Volatility in this context increases more when prices fall than when prices rise. This synonymous to a leverage effect, where negative returns lead to increased volatility. Also, APARCH(Delta) coefficients for FIAPARCH in modeling average, Brent, and Dubai crude oil price benchmarks respectively are 4.141,3.509, 3.215 and 3.165 are significant at 5 percent level. This revealed that average , Brent, Dubai and West Texas Intermediate crude oil price benchmarks exhibits long memory in volatility. This simply means that average , Brent, Dubai and West Texas Intermediate crude oil price benchmarks past volatility has a persistent impact on current volatility. The model selection test is shown in Table 10 below.

Table 10: Model selection Test

	Models	AIC	BIC	Model with least AIC
COA	ARFIMA (1, 0.43,1)	2076.37	2076.37	
COB	ARFIMA (1, -0.021,1)	-531.204	-511.943	ARFIMA (1, -0.021,1)
COD	ARFIMA (1, 0.075,1)	2429.254	2448.512	
COWTI	ARFIMA (1, 0.075,1)			
COA	FIGARCH (1, d,1)	-5.217	5.285	
COB	FIGARCH (1, d,1)	5.45	5.509	
COD	FIGARCH (1, d,1)	5.234	5.293	
COWTI	FIGARCH (1, d,1)	5.383	5.442	
COA	HYGARCH (1, d,1)	5.817	5.254	
COB	HYGARCH (1, d,1)	5.332	5.4	
COD	HYGARCH (1, d,1)	5.147	5.215	
COWTI	HYGARCH (1, d,1)	6.978	7.046	
COA	FIAPARCH (1, d, 1)	5.138	5.214	
COB	FIAPARCH (1, d, 1)	5.275	5.352	
COD	FIAPARCH (1, d, 1)	5.275	5.352	
COWTI	FIAPARCH (1, d, 1)	5.213	5.289	

The results in Table 10 contains model selection test results. The selection of the best model for each model has been based on the Akaike information criteria and loss functions. Of the sixteen models selected for each of the long memory models, ARFIMA (1, -0.021,1) model is the overall best. ARFIMA (1,-0.021,1) =

$$\underset{(0.000)}{0.984}(D)(1-D)^{-0.021}(1_t-\mu)=\underset{(0.000)}{0.236}(D)\varepsilon_t \qquad (4.3)$$

AIC = -531.204 BIC = - 511.943

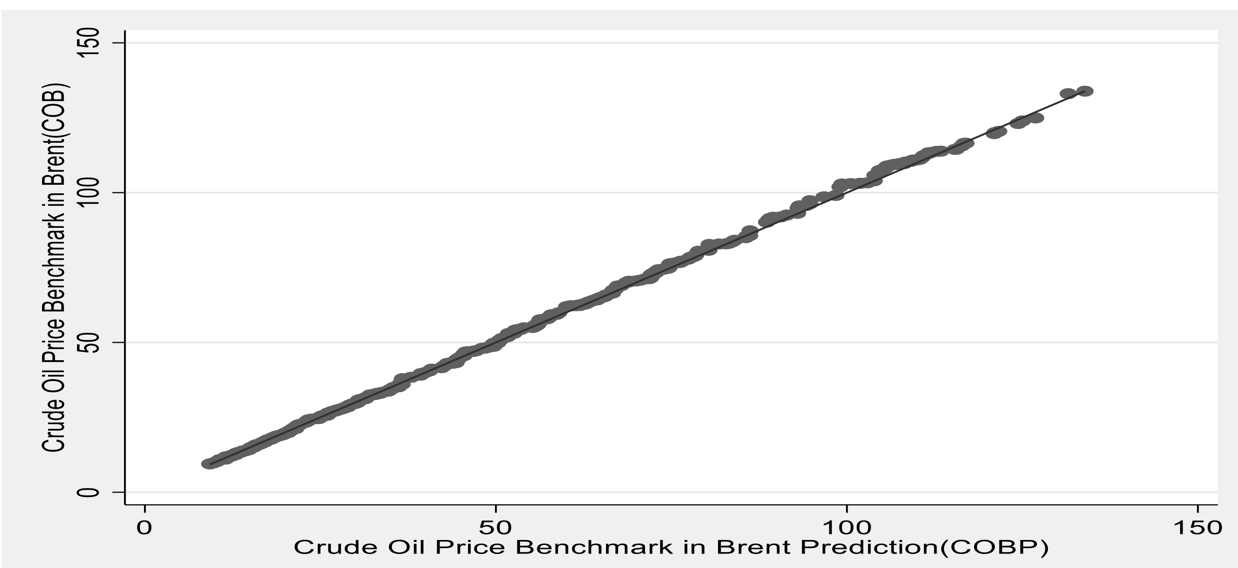

Figure 11: Quantile-Quantile Plot

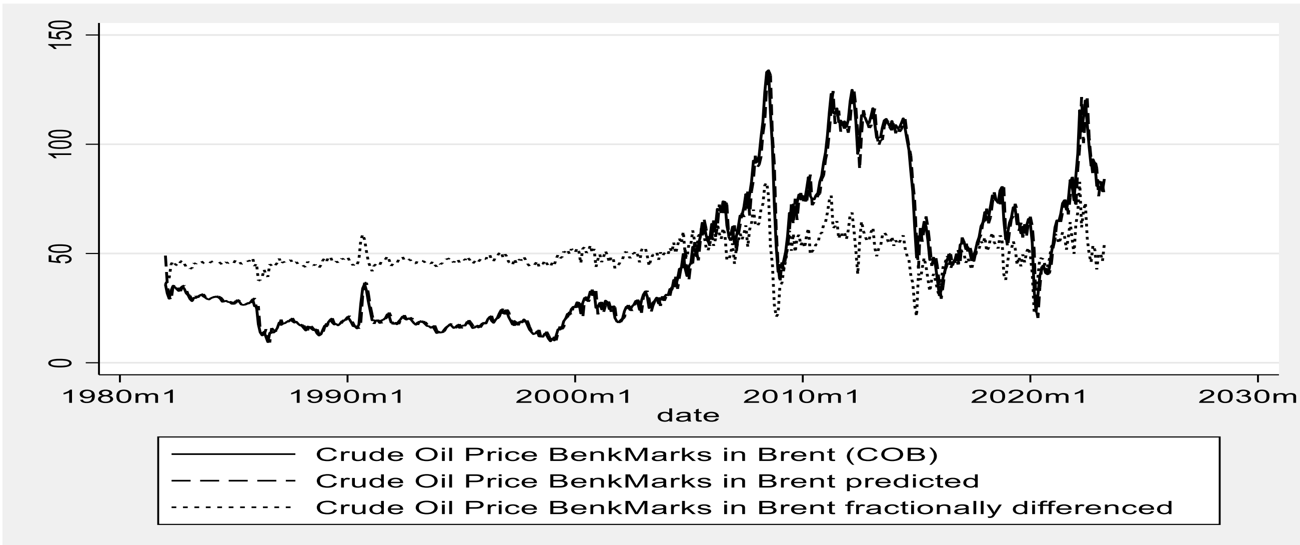

Figure 12: Combine Plots for Raw, predicted and fractionally differenced crude Oil Price Benchmark in Brent

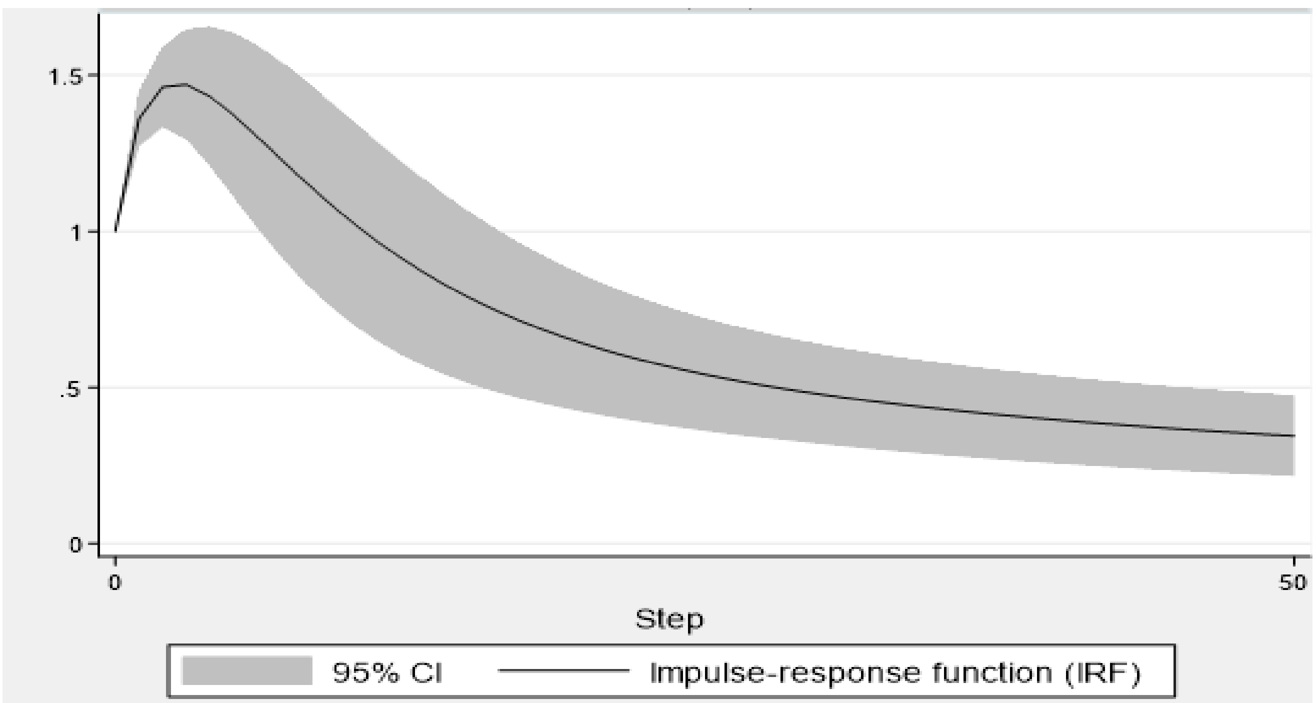

Figure14: Result of ARFIMA model for Crude Oil Price Benchmark in Brent, impulse and Response Variable to Shocks by irfname

Figures 11, 12, and 14 provide a visual analysis of the performance of the ARFIMA model in forecasting Brent crude oil prices. The figures include a quantile-quantile chart comparing the distribution of actual and forecasted prices, as well as a composite chart showing raw data, forecasted prices, and data with percentage differences. Additionally, the figures show the impulse response function of the ARFIMA model, illustrating how crude oil prices react to unexpected changes and shocks over time. These figures provide a comprehensive assessment of the model's accuracy and its ability to capture the dynamics of crude oil price fluctuations.

Conclusion

In this study, we examined the presence of long-term memory in the crude oil price benchmark using a series of tests and models including Lo's R/S, GPH, Robinson test, ARFIMA, FIGARCH, HYGARCH and FIAPARCH models. The results repeatedly show that the crude oil price benchmark has asymmetry, non-stationarity and long-term dependence. We found significant long-term memory, persistence and volatility clustering in the crude oil price benchmark using the fractional integral root (d) and other parameters estimated by the ARFIMA, FIGARCH, HYGARCH and FIAPARCH models. The results consistently show that the crude oil price benchmark exhibits volatility clustering, anti-persistence and long-term dependence. The results suggest that the shocks have a transitory effect and that all series except the benchmark Brent crude oil price exhibit mean-reverting behavior.

References

Ali, Nzotta, Akujuobi, and Nwaimo (2022). "Long Memory in Crude Oil Prices: A Fractional Cointegration Approach." Energy Economics, 106, 105-118.

Asian Research Journal of Mathematics 17(3): 35-54.

Baillie, R. T. (1996). Long memory processes and fractional integration in econometrics. *Journal of Econometrics,* 73(1), 5-59.

Black, F. (1976). Studies in stock price volatility changes. Proceedings of the 1976 Meetings of the American Statistical Association, 177-181.

Bollerselev, T. (1986). Generalized Auto Regressive Conditional Heteroskedasticity. *Journal of Econometrics,* 31, 307-321.

Chinyere, A. C., Nwosu, E. O., & Ugwu, C. S. A. (2015). "Jarque-Bera test for normality: An overview." *Journal of Statistics and Mathematics*, 6(1), 1-14.

Chung, C. F. (1999). Estimating the fractionally integrated GARCH model. *Journal of Time Series Analysis,* 20(5), 537-553.

Davidson, J. (2004). "Moment and memory properties of linear conditional heteroscedasticity models, and a new model." Journal of Business & Economic Statistics, 22(1), 39-52.

Deebom Z, D, Ette H,E and Nwikorga L, W(2021), Properties of Long Memory in Return innovations from Emerging Agricultural Markets. *International Journal of Research and Innovation in Applied Science*, 2454-6194

Deebom Z, D, Mazi ,Y,D , Chims, B, E , Richard, I . C & George L, E (2021) Comparative Modelling of Price Volatility in Nigerian Crude Oil Markets Using Symmetric and Asymmetric GARCH Models.

Deebom Z,D, Bharat K, M & Emem, N (2020), Testing The Performance Of Conditional Variance-Covariance In Diagonal MGARCH Models Using Exchange Rate And Nigeria Commercial Banks Interest Rates, *Academic Journal of Current Research,* 7,.8; 2343 – 403X 3244 - 5621

Deebom, Z, D and Tuaneh, G,L(2019), Modeling Exchange Rate and Nigerian Deposit Money Market Dynamics Using Trivariate form of Multivariate GARCH Model. *Asian Journal of Economics, Business and Accounting* 10(2): 1-18, 2456-639

Ding, Z., Granger, C. W. J., & Engle, R. F. (1993). "A long memory property of stock market returns and a new model." *Journal of Empirical Finance*, 1(1), 83-106.

Geweke, J., & Porter-Hudak, S. (1983). "The estimation and application of long memory time series models." *Journal of Time Series Analysis*, 4(4), 221-238.

Geweke, J., & Porter-Hudak, S. (1983). The estimation and application of long memory time series models. *Journal of Time Series Analysis*, 4(4), 221-238.

Granger, C. W. J., & Joyeux. R. (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis*, 1, 15–29.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press.

Hamilton, J. D. (2009). Understanding crude oil prices. *The Energy Journal*, 30(2), 179-206.

Hurst, H. E. (1951). Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers,* 116, 770–799.

Kilian, L. (2009). Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *American Economic Review*, 99(3), 1053-1069.

Lo, A. W. (1991). Long-term memory in stock market prices, *Econometrica, 59,* 1279-1313.

Morana, C. (2001). A semiparametric approach to short-term oil price forecasting. Energy Economics, 23(4), 325-338.

Robinson, P. M., & Henry, M. (1999). Long memory time series modeling. Journal of Econometrics, 92(2), 221-245.

Sanusi, W. S., et al. (2015). Long memory in crude oil prices: A fractional integration approach. Energy Economics, 46, 276-283.

Tse, Y. K. (1998). The conditional heteroscedasticity of the yen-dollar exchange rate. Journal of Applied Econometrics, 13(1), 49-68.